Proof of Orthogonality Relations
part 2

$\varphi: G \to GL(V)$, $\psi: G \to GL(W)$ irreducible.

$\varphi \not\sim \psi$

Show: $\langle \chi_\psi, \chi_\varphi \rangle = 0$.

WLOG: $\varphi: G \to GL_n(\mathbb{C})$, $\psi: G \to GL_m(\mathbb{C})$

Recall: $\chi_\varphi(g^{-1}) = \overline{\chi_\varphi(g)}$

$$\langle \chi_\psi, \chi_\varphi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\psi(g) \chi_\varphi(g^{-1})$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j} \psi_{ii}(g) \varphi_{jj}(g^{-1})$$

$1 \leq i \leq m$, $1 \leq j \leq n$

$$= \sum_{i,j} \underbrace{\frac{1}{|G|} \sum_{g \in G} \psi_{ii}(g) \varphi_{jj}(g^{-1})}_{0}$$

Lemma: $\varphi: G \to GL(V)$, $\psi: G \to GL(W)$ irreducible, $\varphi \not\sim \psi$

$T \in \mathrm{Hom}(V, W)$. Define

$$T' := \frac{1}{|G|} \sum_{g \in G} \psi_g T \varphi_{g^{-1}}$$

Then $T' = 0$

Schur.

Proof: $T' \in \mathrm{Hom}_G(\varphi, \psi) \simeq \{0\}$

$\varphi: G \to GL_n(\mathbb{C})$, $\psi: G \to GL_m(\mathbb{C})$

$$\begin{array}{ccc} T \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m) & \longleftrightarrow & X = (x_{ij}) \in \mathrm{Mat}_{m \times n}(\mathbb{C}) \\ \Downarrow & & \Downarrow \\ T' = 0 & & X' = 0 \end{array}$$

so

$$X' = \frac{1}{|G|} \sum_{g \in G} \psi_g X \varphi_{g^{-1}} \qquad X = (x_{ij})$$

$= (x'_{ij})$

matrix entries

$$x'_{ij} = \frac{1}{|G|} \sum_{g \in G} \sum_{u,v} \psi_{iu}(g) x_{uv} \varphi_{vj}(g^{-1})$$

$0 =$

$1 \le u \le m, \quad 1 \le v \le n$

$$\Rightarrow \quad 0 = \sum_{u,v} \left[\frac{1}{|G|} \sum_{g \in G} \psi_{iu}(g) \varphi_{vj}(g^{-1}) \right] x_{uv}$$

$X = (x_{uv})$ arbitrary. Plug in $x_{uv} = 1$, others 0

$$\boxed{0 = \frac{1}{|G|} \sum_{g \in G} \psi_{iu}(g) \varphi_{vj}(g^{-1})}$$

$1 \le i, u \le m, \quad 1 \le j, v \le n$

Special case: $i = u, \; j = v$

$$\boxed{0 = \frac{1}{|G|} \sum_{g \in G} \psi_{ii}(g) \varphi_{jj}(g^{-1})}$$

$$\langle \chi_\psi, \chi_\varphi \rangle = \sum_{i,j}{}' \frac{1}{|G|} \sum_{g \in G}{}' \psi_{ii}(g)\, \varphi_{jj}(g^{-1})$$

$$= \sum_{i,j}{}' 0 = 0$$ ✓